ON THE STABILITY OF STEADY MOTIONS

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The question of the stability of steady motion of holonomic mechanical systems with cyclic coordinates has been studied in many books and papers (for example, [1 to 9]) but the subject evidently cannot be considered exhausted. In 1957 Ishlinskii [10] published an example of bifurcation of steady motions which did not lead to unstable modes. A second, similar example, has recently been given by Neimark and Fufaev [11]. Noting the 'unusualness' of the bifurcation the authors of [11] assert that it is associated with the existence of a set of steady motions.

In what follows we consider the stability of steady motions of holonomic mechanical systems; use is made of the theorems of Routh, Poincaré, Kelvin and Chetaev and some new results are obtained. By way of illustration an example is considered, taken from [11]. It is shown that with the proper choice of parameters in this example no singularities occur.

The method evolved in the paper is characterized by a unique approach to the study of the stability of motion of various different mechanical systems and enables us to obtain relatively easily the necessary and sufficient conditions for 'secular' (in a particular sense) stability of steady motion.

1. Consider a system of material points with ideal geometrical constraints under the action of potential forces. Let q_j $(j = 1, \ldots, n)$ denote the independent Langrangian coordinates of the system; $q_j^{\cdot}$ and p_j generalized velocities and impulses; T and Π kinetic and potential energies of the system and $L = T - \Pi$ a Lagrange function.

As fundamental variables which define the state of state of the system at any instant of time t, we use the Routh variables [9]

$$q_i, q_i^{\cdot} \quad (i = 1, \ldots, k), \qquad q_\alpha, p_\alpha \quad (\alpha = k+1, \ldots, n)$$

We introduce the Routh function

$$R(q_i, q_i^{\cdot}, q_\alpha, p_\alpha) = L - \sum_{\alpha=k+1}^{n} p_\alpha q_\alpha \tag{1.1}$$

We assume that this function is not explicitly dependent on time, and write the equations of motion for the system in the form of Routh's equations

$$\frac{d}{dt}\frac{\partial R}{\partial q_i^{\cdot}} - \frac{\partial R}{\partial q_i} = 0 \qquad (i = 1, \ldots, k) \tag{1.2}$$

$$\frac{dq_\alpha}{dt} = -\frac{\partial R}{\partial p_\alpha}, \qquad \frac{dp_\alpha}{dt} = \frac{\partial R}{\partial q_\alpha} \qquad (\alpha = k+1, \ldots n) \tag{1.3}$$

Equations (1.2) and (1.3) admit the energy integral

$$H = \sum_{i=1}^{k} q_i^{\cdot} \frac{\partial R}{\partial q_i^{\cdot}} - R = \text{const} \tag{1.4}$$

Suppose that the coordinates q_α are cyclic, i.e. that $\partial R \,/\, \partial q_\alpha = 0$. Then we immediately obtain from Equations (1.3) the first integrals

$$p_\alpha = c_\alpha \qquad (\alpha = k+1, \ldots, n) \tag{1.5}$$

where c_α are arbitrary constants of integration. Thus Equations (1.2) contain only the variables q_i together with their first and second derivatives with respect to time t, and the constants c_α. These equations can be looked upon as the equations of motion of a system with k degrees of freedom, characterized by the Lagrangian function $R\ (q_i,\ q_i^{\cdot},\ c_\alpha)$. We shall call this a reduced system corresponding to the initial mechanical system and characterized by the Lagrangian function $L\ (q_i,\ q_i^{\cdot},\ q_\alpha^{\cdot})$. After system (1.2) has been integrated the cyclic coordinates can be found from Equations (1.3) in the form of quadratures.

By Expression (1.1) the Routh function is

$$R = R_2 + R_1 - W$$

in view of which Equations (1.2) assume the form

$$\frac{d}{dt}\frac{\partial R_2}{\partial q_i^{\cdot}} - \frac{\partial R_2}{\partial q_i} = -\frac{\partial W}{\partial q_i} + \sum_{j=1}^{k} g_{ij} q_j^{\cdot} \qquad (i = 1, \ldots, k) \tag{1.6}$$

Here

$$R_2 = \frac{1}{2}\sum_{i,\,j=1}^{k} a_{ij}(q)\, q_i^{\cdot} q_j^{\cdot}, \qquad W = \Pi(q) + \frac{1}{2}\sum_{\alpha,\,\beta=k+1}^{n} b_{\alpha\beta}(q)\, p_\alpha p_\beta$$
$$R_1 = \sum_{i=1}^{k} a_i(q,\, c)\, q_i^{\cdot} \tag{1.7}$$

denote respectively the kinetic and potential energies of the reduced system and the part of the Routh function which is linear in the velocities $q_i^{\cdot}$ and

$$g_{ij} = \frac{\partial a_j}{\partial q_i} - \frac{\partial a_i}{\partial q_j}, \qquad g_{ij} = -g_{ij}$$

are gyroscopic coefficients. The function R_2 will be the positive definite quadratic form of the velocities $q_i^{\cdot}$ $(i = 1, \ldots, k)$.

Thus if the initial system is under the action of potential forces, derivatives of the force function Π, then the reduced system is under the action of potential forces, derivatives of a force function W, and gyroscopic forces. The latter will not exist only when all the gyroscopic coefficients $g_{ij} = 0$ $(i, j = 1, \ldots, k)$. In this case the system is called gyroscopically unconstrained. We further assume that the function W is a continuous function of the coordinates q_i with continuous partial derivatives with respect to q_i.

2. Under certain initial conditions q_i°, $q_i^{\cdot\circ} = 0$, p_α° systems with cyclic coordinates under the action of potential forces can undergo steady motion in which the positional

coordinates q_i and the velocities of the cyclic coordinates q_α retain their initial values and cyclic coordinates vary linearly with time. For fixed values $p_\alpha{}^\circ = c_\alpha$ the constants $q_i{}^\circ$ can be found from the equations

$$\partial W / \partial q_i = 0 \qquad (i = 1, \ldots, k) \tag{2.1}$$

which are the equations of equilibrium of the reduced system.

The potential energy $W(q_i, c_\alpha)$ of the latter depends not only on the positional coordinates q_i, but also on the arbitrary constants c_α $(\alpha = k+1, \ldots, n)$, which can be looked upon as parameters. A general theory of the equilibrium of such systems for various values of the parameters has been proposed by Poincaré [3 and 5].

Let

$$q_i = \varphi_i{}^{(s)}(c_{k+1}, \ldots, c_n) \qquad (i = 1, \ldots, k;\ s = 1, 2, \ldots) \tag{2.2}$$

represent the roots of Equations (2.1); we assume that the functions $\varphi_i{}^{(s)}(c_\alpha)$ are continuous functions of parameters c_α. Thus the steady motions form sets of dimension $n - k$, each point of which is a stationary point of the potential energy W for fixed c_α.

In the n-dimensional space G of the variables q_i and c_α Equations (2.2) define hypersurfaces c_s which together form some real hypersurface B, different points of which correspond to possible states of steady motion.

The roots (2.2) of Equations (2.1) are determined uniquely at all ordinary points of the space G where the Hessian of the potential energy of the reduced system

$$\Delta = \left\| \frac{\partial^2 W}{\partial q_i \partial q_j} \right\| \neq 0 \tag{2.3}$$

Points at which $\Delta = 0$ are called critical or bifurcation points; in the neighbourhood of such points Equations (2.1) do not have a unique solution. At such points different hypersurfaces C_s may intersect or have tangential hyperplanes orthogonal to the subspace C_α^π of the space G.

Consider the question of stability in the Liapunov sense of some steady motion corresponding to a specific point on the surface B. Without loss of generality we can assume that for fixed values of c_α this steady motion corresponds to zero values of the function W and all the positional coordinates $q_i = 0$.

Let us now apply to the system sufficiently small initial perturbations, assuming that in the perturbed motion

$$q_\alpha = q_\alpha{}^\circ + \xi_\alpha, \qquad p_\alpha = p_\alpha{}^\circ + \eta_\alpha$$

and retaining the previous notations for the positional coordinates and their velocities. The equations of the perturbed motion will now be given by Equations (1.6) and the equations obtained from (1.3). In explicit form these equations are as follows:

$$\sum_{j=1}^{k} a_{ij}(q_s) q_j^{\cdot\cdot} + \sum_{s,\, r=1}^{k} \left(\frac{\partial a_{is}}{\partial q_r} - \frac{1}{2} \frac{\partial a_{rs}}{\partial q_i} \right) q_r^{\cdot} q_s^{\cdot} - \tag{2.4}$$

$$- \sum_{j=1}^{k} g_{ij}(q_s, p_\alpha) q_j^{\cdot} + \frac{\partial W(q_s, p_\alpha)}{\partial q_i} = 0 \qquad (i = 1, \ldots, k) \tag{2.5}$$

$$\frac{d\xi_\alpha}{dt} = -\frac{\partial R}{\partial \eta_\alpha} - q_\alpha^{\cdot\circ}, \qquad \frac{d\eta_\alpha}{dt} = 0 \qquad (\alpha = k+1, \ldots\, n)$$

From the first group of equations of (2.5) we find that steady motion is unstable with respect to cyclic coordinates q_α in the general case of $q_\alpha^{\cdot\circ} \neq -\partial R / \partial\eta_\alpha$. Consequently, it is reasonable to speak of the instability of steady motion only with respect to the quantities q_i, $\dot q_i$ and p_α or to some continuous single-valued bounded functions of these quantities.

The equations of perturbed motion have, evidently, the energy integral

$$H = R_2 + W = \text{const} \tag{2.6}$$

and also the cyclic first integrals

$$\eta_\alpha = \text{const} \tag{2.7}$$

Due to the existence of integrals (2.7) motion of a system with cyclic coordinates is obviously stable with respect to p_α.

With arbitrary sufficiently small perturbations the equations of perturbed motion will therefore be given by Equations (2.4) for the reduced system near its equilibrium position, and they will depend on fixed values of $\eta_\alpha \neq 0$.

But in this case it may happen that

$$\left(\frac{\partial W(q_s, c_\alpha + \eta_\alpha)}{\partial q_i}\right)_{q_i=0} \neq 0$$

i.e. the equilibrium position $q_i = 0$ of the reduced system will not satisfy Equations (2.4) and its perturbed motion can be accomplished under constantly acting perturbations, which belong to the category of potential perturbations ([5], p. 255). In order not to consider these perturbations we can set, as is usually done, all $\eta_\alpha = 0$, i.e. confine our attention to a study of stability under perturbations which do not alter the constants c_α. Stability of this sort is called Liapunov conditional stability.

Note that the supposition of the imperturbability of cyclic impulses $p_\alpha = c_\alpha$ indicates only that to every perturbed motion of the system there corresponds a definite steady motion [7]. In fact, if we impart to the unperturbed motion, for which $p_\alpha = c_\alpha$, arbitrary sufficiently small perturbations, we can consider the perturbed motion of the system as described by Equations (2.4) for fixed values of $\eta_\alpha \neq 0$, as perturbed motion corresponding to steady motion for which $p_\alpha = c_\alpha + \eta_\alpha$, and $q_i = q_{i0}'$, and

$$\left(\frac{\partial W(q_i, c_\alpha + \eta_\alpha)}{\partial q_i}\right)_{q_i=q_{i0}'} = 0$$

where we can make $|q'_{i0}|$ as small as we like.

It is well known that the stability of steady motion can be effectively investigated by using Routh's theorem [1], which can be formulated as a particular case of a corollary of Liapunov's stability theorem ([5], p. 19). According to this theorem if for given $p_\alpha = c_\alpha$ the energy integral H = const has an isolated minimum, then the steady motion is stable, at any rate for perturbations which do not alter the values of the integrals $p_\alpha = c_\alpha$.

Liapunov [4] observed that if the integral $H=$ const has a minimum not only for the given values of $p_\alpha = c_\alpha$ but also for any sufficiently close to them $p_\alpha = c_\alpha + \eta_\alpha$, and if the values of the variables q_i which make H a minimum are continuous functions of p_α, then the steady motion is stable for any perturbation.

This theorem can easily be proved by the method of normal ϵ-proofs. We shall not, however, dwell upon this, but simply observe that geometrically the theorem is almost obvious. Since R_2 is positive definite the integral H has a minimum when and only when the function W has a minimum. If the unperturbed motion corresponds to some point on the surface B which does not coincide with the critical point but corresponds to the isolated (for the given $p_\alpha = c_\alpha$) minimum of the function W, then for sufficiently small perturbations $\eta_\alpha \neq 0$ the steady motion remains all the time in the neighbourhood of the minimum of W corresponding to the values $p_\alpha = c_\alpha + \eta_\alpha$ and is absolutely stable.

Since Routh's theorem in effect amounts to Lagrange's theorem for the reduced system there exists a well-known analogy between steady motion and the equilibrium of holonomic systems under the action of potential forces. In general, in problems of instability no similar analogy exists, since the steady motion may be stable also when the function W has no minimum – in such cases gyroscopic stabilization takes place [5].

Consequently we can talk of the converse of Routh's theorem when the function W has no minimum only on certain supplementary conditions. The most simple case of the converse of Routh's theorem is a gyroscopically unconstrained system on which no gyroscopic forces act. In this case the theorems of Liapunov and Chetaev [5] on the converse of Lagrange's theorem are fully applicable.

In the case of gyroscopically constrained system the problem becomes more complex. One particular case of the converse of Routh's theorem is the well-known theorem of Kelvin, according to which it can be said that if the number of negative roots of the equation

$$\Delta(\lambda) = \| c_{ij} - \delta_{ij}\lambda \| = 0, \qquad c_{ij} = \left(\frac{\partial^2 W}{\partial q_i \partial q_j}\right)_0, \qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.8}$$

is odd or if there are no zero roots, then the motion is unstable. If the number of negative roots is even, gyroscopic stabilization is possible.

We indicate a further particular case of the converse of Routh's theorem. Suppose that for the steady motion $q_j = 0$ function W has no minimum and, that for values of the variables q_j arbitrarily small in absolute magnitude it can assume negative values. Consider the function

$$V = -H \sum_{i=1}^{k} q_i \frac{\partial R}{\partial q_i^{\cdot}}$$

In the neighbourhood of values of q_i and $q_i^{\cdot}$ arbitrarily small in absolute magnitude we isolate a region defined by the compatible inequalities

$$H < 0, \qquad \sum_{i=1}^{k} q_i \frac{\partial R}{\partial q_i^{\cdot}} > 0 \tag{2.9}$$

From the equations of perturbed motion (1.2) the derivative of the function V with respect to time can be expressed in the form

$$V^{\cdot} = -H\left(2R_2 + R_1 + \sum_{i=1}^{k} q_i \frac{\partial R}{\partial q_i}\right) \tag{2.10}$$

On the basis of Chetaev's instability theorem we conclude that the following theorem is valid.

Theorem 2.1. If in the region (2.9) the function

$$2R_2 + R_1 + \sum_{i=1}^{k} q_i \frac{\partial R}{\partial q_i} \tag{2.11}$$

is positive definite with respect to q_i and $q_i^{\cdot}$, then the steady motion is unstable.

Corollary 2.1. If in the region (2.9) the quadratic part $R^{(2)}$ of the expansion of Routh's function into a Maclaurin series

$$R = R^{(2)} + \ldots \tag{2.12}$$

is positive definite, then the steady motion is unstable [12].

Indeed, substituting (2.12) into (2.10), we have

$$V^{\cdot} = -H(2R_2^{(2)} + 2R_1^{(2)} - 2W^{(2)} + \ldots) = -H(2R^{(2)} + \ldots)$$

which proves the assertion. Here

$$R_1^{(2)} = \sum_{ij=1}^{k} \left(\frac{\partial a_i}{\partial q_j}\right)_0 q_j q_i^{\cdot}, \qquad W^{(2)} = \frac{1}{2}\sum_{ij=1}^{k} c_{ij} q_i q_j$$

the row of dots representing infinitesimals of order higher than the second.

Remark 2.1. If $R^{(2)}$ is positive definite in any sufficiently small neighbourhood of $q_i = 0$, it can be shown that the above corollary is equivalent to the theorem in Section 4 of [13].

Remark 2.2. Other particular cases of the converse of Routh's theorem are shown in [14].

To conclude this section we note that for a specific hypersurface C_s of steady motions (2.2) the Hessian of potential energy (2.3) is given by

$$\Delta = \Delta(c_{k+1}, \ldots, c_k)$$

Therefore, loss of stability of steady motion and also of equilibrium can occur only at critical points at which at least one of the roots of Equation (2.8) passes, in changing its sign, through zero [3 and 5].

3. We proceed now to an investigation of the effects of dissipative forces on the stability of steady motion. First consider the case of the dissipative forces

$$Q_i = -\frac{\partial f}{\partial q_i^{\cdot}} \qquad (i = 1, \ldots, k) \tag{3.1}$$

the derivatives of Rayleigh's function

$$2f = \sum_{i,\ j=1}^{k} e_{ij}(q)\, q_i^{\cdot} q_j^{\cdot} \tag{3.2}$$

which we shall assume to be a positive definite function of q_i $(i = 1, \ldots, k)$.

The equations of perturbed motion in this case differ from Equations (2.4) only in the right-hand sides of (3.1). From these equations we easily obtain the equation

$$\frac{d}{dt}(R_2 + W) = -2f \tag{3.3}$$

for the rate of energy dissipation.

Theorem 3.1. If the function W has an isolated minimum for the given values of $p_\alpha = c_\alpha$ and also for any sufficiently close values $p_\alpha = c_\alpha + \eta_\alpha$, and if the values of q_i which make it a minimum are continuous functions of p_α then the dissipative forces (3.1) do not destroy the stability of the steady motion corresponding to the given $p_\alpha = c_\alpha$ and any sufficiently close perturbed motion tends asymptotically to steady corresponding to the values of $p_\alpha = c_\alpha + \eta_\alpha$.

Let us disturb the system in its steady motion under consideration by imparting to the points of this system small initial deviations and velocities. On its own the system will behave in accordance with Equations (3.3), which leads to the inequality

$$R_2 + W(q_s, c_\alpha) + W' \leqslant R_2^{(0)} + W(q_i^\circ, c_\alpha + \eta_\alpha) \tag{3.4}$$

where the superscript $^\circ$ denotes the initial value of the relevant quantity, and

$$W' = W(q_s, c_\alpha + \eta_\alpha) - W(q_s, c_\alpha)$$

By W_1 we denote the smallest possible value which $W(q_s, c_\alpha)$ can take if one of the coordinates q_i becomes equal in absolute magnitude to a given arbitrarily small number $A > 0$ and the remaining positional coordinates $|q_i| \leqslant A$. Obviously $W_1 > W(0, c_\alpha)$. The initial values q_i° of the coordinates q_i can be taken sufficiently small, so that $W^{(0)} < W_1$. Whatever the initial position of the system was, the initial velocities can be taken such that the constants $R_2^{(0)}$, and $|\eta_\alpha|$ are as small as we like. We select these constants to be so small that

$$R_2^{(0)} + W^{(0)} - W' < W_1$$

for all values of $|q_i| \leqslant A$, which is possible with the assumptions made concerning continuity. With this choice of initial conditions during the whole of the subsequent motion we shall have

$$R_2 + W < W_1 \tag{3.5}$$

It follows that $W < W_1$. This inequality is satisfied at least while $|q_i| \leqslant A$. But the initial values of $|q_i^\circ|$ are necessarily less than A, and since q_i vary continuously with time, $|q_i|$ cannot exceed A without first equalling this number. But from (3.5) the equalities $|q_i| = A$ are obviously impossible. Stability with respect to velocities also follows from the inequalities (3.5).

Consequently any perturbed motion of the system which is sufficiently close to the given steady motion will always occur in the arbitrarily small neighbourhood of the unperturbed motion. But according to Equation (3.3) the energy of the system H in perturbed motion is dissipated until all the positional coordinates q_i are constant, and, under the conditions of the theorem, this is possible only for steady motion corresponding to the minimum of W with perturbed values of $p_\alpha = c_\alpha + \eta_\alpha$. Thus the theorem is proved.

Corollary 3.1. If the function W has an isolated minimum for given c_α then the steady motion becomes asymptotically stable when the dissipative forces (3.1) are added, provided the values of the constants $p_\alpha = c_\alpha$ are not altered.

Remark 3.1. For ordinary points on the surface B when the second variation $W^{(2)}$ of the function W is positive definite, the above corollary becomes the theorem of Kelvin [2 and 5].

Remark 3.2. When $W^{(2)}$ is positive definite it can easily be shown that the characteristic equation for the first approximation equations for the system of equations of perturbed motion is of the form [11]

$$D(\lambda) = \lambda^{n-k} D_1(\lambda) = 0 \tag{3.6}$$

where

$$D_1(\lambda) = \left\| \sum_{j=1}^{k} a_{ij}\lambda^2 - \sum_{j=1}^{k} (g_{ij} + e_{ij})\lambda + \sum_{j=1}^{k} c_{ij} \right\| = 0 \tag{3.7}$$

is the characteristic equation for the first approximation of the equations of perturbed motion for $\eta_\alpha = 0$. Since when $W^{(2)}$ is positive definite all the roots of the latter have negative real parts, then for the system of equations of perturbed motion all the conditions of the Liapunov-Malkin [15] stability theorem are satisfied in the special case of several zero roots.

Remark 3.3. The asymptotic stability of steady motion under the conditions of Theorem (3.1) can also be proved by making use of the theorem of Krasovskii ([15], par. 103). In fact in the present case the set $M\,(q_i^{\cdot} = 0\ (i = 1, \ldots, k))$ does not contain whole modes of the system in arbitrarily small neighbourhood of steady motion.

We can also prove [16] a theorem which constitutes the converse of Theorem (3.1) and generalizes the corresponding theorem of Kelvin [2 and 5]:

Theorem 3.2. If for a steady motion, isolated for given c_α, the function W has no minimum and if in the sufficiently small neighbourhood, of this motion it can assume negative values, then the steady motion is unstable.

As is well known [2 and 5], Kelvin introduced the important concept of secular and temporary stability of equilibrium positions. These concepts can evidently be extended to stability of steady motion. In doing so, we shall differentiate between cases when in the perturbed motion the system is subjected to dissipative forces which depend on the velocities only of the positional coordinates (the impulses $p_\alpha = c_\alpha + \eta_\alpha$ remaining constant), and when the system is subjected to dissipative forces which depend on the velocities of all the coordinates. In the first case the stability will be called 'secular' (in quotes) and in the second case, secular (without quotes).

From Theorems (3.1) and (3.2) we obtain the obvious conditions necessary and sufficient for the 'secular' stability of steady motion.

Note also that for 'secular' stability of steady motion as well as for positions of equilibrium the law of loss of stability for fixed values of parameters is valid [3 and 5].

Finally we shall look very briefly at the case when a system with cyclic coordinates is subjected to dissipative forces, derivatives of a Rayleigh function

$$2F(q_i^{\cdot}, q_\alpha^{\cdot}) = \sum_{i,j=1}^{n} e_{ij} q_i^{\cdot} q_j^{\cdot}, \qquad e_{ij} = \text{const} \tag{3.8}$$

which is negative definite with respect to all velocities $q_j^{\cdot}$ $(j = 1, \ldots n)$. In this case steady motion is possible only if certain constant forces R_j are applied to the system which balance the dissipative forces in the state of steady motion under consideration [17]. The equations of motion are then of the form

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \frac{\partial F}{\partial q_i^{\cdot}} + R_i, \qquad \frac{d}{dt}\frac{\partial L}{\partial q_\alpha^{\cdot}} = \frac{\partial F}{\partial q_\alpha^{\cdot}} + R_\alpha \qquad \begin{pmatrix} i = 1, \ldots, k \\ \alpha = k+1, \ldots, n \end{pmatrix} \tag{3.9}$$

whence we obtain an equation for the rate of energy change

$$\frac{d}{dt}(T + \Pi) = 2F + \sum_{j=1}^{n} R_j q_j^{\cdot} \tag{3.10}$$

Equations (3.9) have the obvious solution

$$q_i = q_i^{\cdot} = 0, \qquad q_\alpha^{\cdot} = q_\alpha^{\cdot\circ} \tag{3.11}$$

which describes steady motion if for the values (3.11) the conditions

$$\left(\frac{\partial L}{\partial q_j} + \frac{\partial F}{\partial q_j^{\cdot}}\right)_0 + R_j = 0 \qquad (j = 1, \ldots, n) \tag{3.12}$$

are satisfied.

For the perturbed motion it is not difficult to obtain from Equation (3.10) the equation

$$\frac{d}{dt}[H^{(2)}(q_i, q_i^{\cdot}, \xi_\alpha^{\cdot}) + \cdots] = 2F(q_i^{\cdot}, \xi_\alpha^{\cdot}) \tag{3.13}$$

where the row of dots indicates terms of order not lower than the third.

An exact repeat of the proof given by Chetaev ([5], pp. 77-79) for the theorems of Kelvin shows that if the second variation $H^{(2)}$ of the energy of the system is a positive definite function of the variables q_i, $q_i^{\cdot}$ and $\xi_\alpha^{\cdot}$, then the unperturbed steady motion is asymptotically stable, and if $H^{(2)}$ can assume negative values, then it is unstable [17].

These results have generalizations analogous to Theorems 3.1 and 3.2.

From this follow the obvious conditions which are necessary and sufficient for secular stability of steady motion.

4. *Example.* Consider a heavy solid body with a horizontal axis which can rotate about a vertical passing through the point O of intersection of the axis of a pendulum with a plane orthogonal to this axis and containing the centre of gravity of the pendulum [10 to 12]. We take the point O as the origin of a fixed system of coordinate axes $O\xi\eta\zeta$ with the ζ-axis directed vertically downwards and also as the origin of a moving system of coordinates $Oxyz$, the x-axis of which lies along the horizontal axis of the pendulum, the z-axis along a straight line passing through the centre of gravity of the pendulum and the y-axis orthogonal to these two axes. The angle between the ζ- and z-axes will be denoted by θ and the angle between the ξ- and x-axes by φ. These two angles completely define

the position of the body in space. We shall take the axes of x, y and z as the principal axes of inertia of the body for the point O; the moments of inertia of the pendulum about these axes will be denoted by A, B and C, respectively. Let Mg be the weight of the pendulum and $\dot{a} > 0$ the coordinate of the centre of mass along the z-axis.

In [11] the case is considered when $A = B$ and, in addition to gravity forces, dissipative forces are applied, which depend on $\theta^{\cdot}$. The first integral of the equations of motion

$$p = \frac{\partial L}{\partial \varphi^{\cdot}} = (B \sin^2 \theta + C \cos^2 \theta) \varphi^{\cdot} = c \tag{4.1}$$

corresponds to the cyclic coordinate φ.

We introduce the notations

$$\alpha = \frac{B - C}{B}, \qquad \alpha < 1, \qquad \beta = \frac{c^2}{MgaB}, \qquad \beta \geqslant 0 \tag{4.2}$$

and with the accuracy to an insignificant multiplier we write the potential energy of the reduced system in the form

$$W = \frac{\beta}{2(1 - \alpha \cos^2 \theta)} - \cos \theta \tag{4.3}$$

For this pendulum α is obviously constant.

The equation of steady motion

$$\frac{\partial W}{\partial \theta} = \sin \theta \left(1 - \frac{\alpha\beta \cos \theta}{(1 - \alpha \cos^2 \theta)^2}\right) = 0 \tag{4.4}$$

has the roots $\theta_1 = 0$ and $\theta_2 = \pi$ for any value of the parameter β and another root θ_3 where $(1 - \alpha \cos^2 \theta_3)^2 / (\alpha \cos \theta_3) = \beta$, or $\theta_3 \cos^{-1}(1/\alpha\Omega)$, if $\alpha \neq 1$ and [11]

$$\Omega = \frac{B\varphi^{\cdot 2}}{Mga} \geqslant |\alpha|^{-1}$$

Thus in the $\theta\beta$-plane there are three branches C_s of the curve of steady motion, the first two of which are the straight lines $\theta = 0$ and $\theta = \pi$. Let us examine the shape of the third branch $\theta_3 = \theta_3(\beta)$, given in the form of an implicit function by the equation

$$f(\theta, \beta) = 1 - \frac{\alpha\beta \cos \theta_3}{(1 - \alpha \cos^2 \theta_3)^2} = 0 \tag{4.5}$$

Since $\beta \geqslant 0$ it follows that

$$0 \leqslant \theta_3 \leqslant \pi/2 \quad \text{if} \quad \alpha > 0, \ \pi/2 \leqslant \theta_3 \leqslant \pi, \quad \text{if} \quad \alpha < 0,$$

and

$$\theta_3 = 0, \pi \quad \text{for} \quad \beta = \beta_1 = (1 - \alpha)^2 |\alpha|^{-1} \tag{4.6}$$

Taking the total differential of (4.5) with respect to β we find, that

$$\frac{d\theta_3}{d\beta} = \frac{\alpha \cos^2 \theta_3}{(1 - \alpha \cos^2 \theta_3)(1 + 3\alpha \cos^2 \theta_3) \sin \theta_3}$$

Obviously $d\theta_3 / d\beta = 0$ only for $\theta_3 = {}^1/_2\,\pi$.

Also, $d\theta_3 / d\beta = \infty$ for $\theta_3 = 0, \pi$ and for $\cos^2 \theta_3 = -{}^1/_3\alpha$, provided that $\alpha < -1/3$. The parameter β is then given by

$$\beta_2 = \frac{16}{9}\left(\frac{3}{|\alpha|}\right)^{1/2} \tag{4.7}$$

Within the interval under consideration $0 \leqslant \theta_3 \leqslant \pi$ the sign of the derivative $d\theta_3 / d\beta$ is determined by the sign of $(1 + 3\alpha \cos^2\theta_3)^{-1}$, hence $d\theta_3 / d\beta > 0$, if $\alpha > 0$ or if $\alpha < 0$ but $\cos^2\theta_3 > -1/(3\alpha)$; if $\alpha < 0$ and $\cos^2\theta_3 < -1/(3\alpha)$, then $d\theta_3 / d\beta < 0$. Note also that $\theta_3 \to {}^1/_2\pi$ as $\beta \to \infty$.

The approximate shape of the curve $\theta_3 = \theta_3(\beta)$ for various values of α is shown in the figure.

The Poincaré stability coefficient for the reduced system is

$$\delta_s \equiv \left(\frac{\partial^2 W}{\partial\theta^2}\right)_s = \cos\theta_s\left(1 - \frac{\alpha\beta\cos\theta_s}{(1-\alpha\cos^2\theta_s)^2}\right) + \alpha\beta\sin^2\theta_s\frac{1+3\alpha\cos^2\theta_s}{(1-\alpha\cos^2\theta_s)^3}$$

where θ_s denotes one of the three values of the roots of Equation (4.4). For the branch C_1, $\theta = \theta_1 = 0$, so that

$$\delta_1 = 1 - \alpha\beta\,(1-\alpha)^{-2}$$

If $\alpha > 0$, then $\delta_1 > 0$ for $\beta < \beta_1$, and $\delta_1 < 0$ for $\beta > \beta_1$. If $\alpha < 0$, then $\delta_1 > 0$ for any value of $\beta \geqslant 0$.

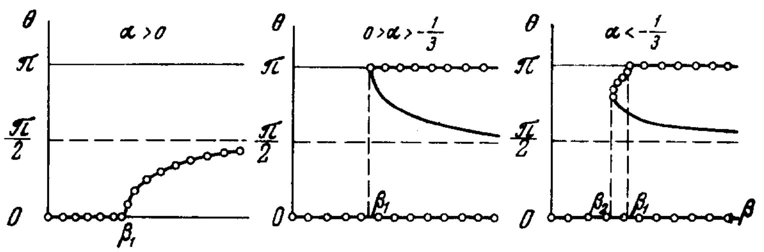
For the branch C_2, $\theta = \theta_2 = \pi$, so that

$$\delta_2 = -\,[1 + \alpha\beta\,(1-\alpha)^{-2}]$$

If $\alpha > 0$ then $\delta_2 < 0$ for any value of $\beta \geqslant 0$; if $\alpha < 0$ then $\delta_2 > 0$ for $\beta > \beta_1$ and $\delta_2 < 0$ for $\beta < \beta_1$. For the third branch C_3, when $\theta = \theta_3$, we have

$$\delta_3 = \alpha\beta\,\sin^2\theta_3\,(1 + 3\alpha\cos^2\theta_3)\,(1 - \alpha\cos^2\theta_3)^{-3}$$

If $\alpha > 0$, then $\delta_3 > 0$ for any value of $\beta > 0$; if $0 > \alpha \geqslant -1/3$ then $\delta_3 < 0$ for any value of β.



If $\alpha < -1/3$ then $\delta_3 > 0$ for $\cos^2\theta_3 > -1/3\alpha$ and $\beta_2 < \beta < \beta_1$, and $\delta_3 < 0$ for $\cos^2\theta_3 < -1/3\,\alpha$.

Consequently it can be asserted that the steady motions of the pendulum are:

(A) 'secularly' stable if they correspond to points:

(1) on the branch C_1 for any value of $\beta \geqslant 0$ and $\alpha < 0$ or for $\beta < \beta_1$, and $\alpha > 0$;

(2) on the branch C_2 for $\beta > \beta_1$ and $\alpha < 0$;

(3) on the branch C_3 for any value of β, if $\alpha > 0$ and for* $\beta_2 < \beta < \beta_1$ if $\alpha < -1/3$ and $\cos^2\theta_3 > -1/3\alpha$;

(B) unstable if they correspond to points:

(1) on the branch C_1 for $\beta > \beta_1$ and $\alpha > 0$;

(2) on the branch C_2 for $\beta < \beta_1$ and $\alpha < 0$ or for any value of β if $\alpha > 0$;

(3) on the branch C_3 for any value of β if $0 > \alpha \geqslant -1/3$ or if $\alpha < -1/3$ and $\cos^2\theta_3 > -1/3\alpha$;

In the figure the regions of stability are denoted by small circles. Points M_1 $(\theta = 0,\ \beta = \beta_1)$ and M_2 $(\theta = \pi,\ \beta = \beta_1)$ are bifurcation points; the point M_3 $(\theta = \theta_3,\ \beta = \beta_2)$ is also critical and loss of stability on the corresponding branches takes place at these points. Note also that the tangents to the branch C_3 at the points M_i are parallel to the axis $\beta = 0$, i.e. these points are limiting point [6] for the branch C_3. It is also easy to check that the law of loss of stability is satisfied for fixed values of the parameter β.

In [11] a figure is given (Fig. 2) for the case of $A = B$, $-1 < \alpha < -1/3$, which shows curves of steady motions in the $\theta\,\Omega$-plane. For $|\alpha|^{-1} < \Omega < \sqrt{3|\alpha|^{-1}}$ steady motions corresponding to points on all three branches are found to be stable, which would appear to contradict the law of loss of stability for a fixed value of Ω and causes, as it were, an 'unusual' nature of bifurcation. In fact the curves should be drawn in the $\theta\beta$-plane (see Figure) since the constant parameter here is β, not Ω.

Note that the bifurcation of the kind considered here is encountered, for example, in the theory of equilibrium curves for a rotating liquid [6].

We consider finally the question of the stability of steady motion of a pendulum when, apart from gravity, it is subjected to dissipative forces with total dissipation and constant forces which balance the dissipative forces in the steady state.

With the notations of (4.2), we find that

$$\frac{\partial^2 W}{\partial c^2} = \frac{1}{1-\alpha\cos^2\theta}, \qquad \frac{\partial^2 W}{\partial\theta\,\partial c} = -2\alpha c\,\frac{\sin\theta\cos\theta}{(1-\alpha\cos^2\theta)^2}$$

It can easily be seen that the sign of

$$\left(\frac{\partial^2 W}{\partial c^2}\right)_s \delta_s - \left(\frac{\partial^2 W}{\partial\theta\,\partial c}\right)_s^2$$

coincides with the sign of δ_s for $\theta = \theta_1 = 0$ and $\theta = \theta_2 = \pi$ and with the sign of α for $\theta = \theta_3$.

Consequently, under the action of dissipative forces with total dissipation and of constant forces balancing the dissipative, steady motions corresponding to points: on the branch C_1 for any value of $\beta \geqslant 0$ and $\alpha < 0$ or for $\beta < \beta_1$ and $\alpha > 0$, on the branch C_2 for $\beta > \beta_1$ and $\alpha < 0$ and on the branch C_3 for $\alpha > 0$ are secularly stable, and those corresponding to all other points on the branches C_s are unstable.

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* Note that the assertion made in the last two lines of [12] is valid with the exception of this case.

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